The Vasicek Interest Rate Process
Part III - Zero Coupon Bond Price Equation

Gary Schurman, MBE, CFA

March, 2013

In Part III of the Vasicek Interest Rate Process series we will calculate the price of a pure discount bond that was purchased at time s and matures at time t.

The Stochastic Discount Rate Mean and Variance

We will define the variable m to be the mean of the stochastic discount rate $R_{s,t}$ over the time interval $[s,t]$. In Part II we determined that the equation for the mean of the stochastic discount rate was... [2]

$$m = r_\infty (t-s) + (r_\infty - r_s)\left(\mathrm{Exp}\left\{-\lambda (t-s)\right\} - 1\right)\lambda^{-1} \tag{1}$$

We will define the variable v to be the variance of the stochastic discount rate $R_{s,t}$ over the time interval $[s,t]$. In Part II we determined that the equation for the variance of the stochastic discount rate was... [2]

$$v = \frac{\sigma^2}{2\lambda^3}\left(2\lambda(t-s) - 3 + 4\,\mathrm{Exp}\left\{-\lambda(t-s)\right\} - \mathrm{Exp}\left\{-2\lambda(t-s)\right\}\right) \tag{2}$$

In Equations (1) and (2) above r_s is the short rate at time s, r_∞ is the long-term short rate, λ is the rate of mean reversion, and σ is the annual short rate volatility.

Our Hypothetical Problem

The go-forward interest rate assumptions from Parts I and II were...

Description	Symbol	Value
Short rate at time zero	r_0	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: What is the mean and variance of the stochastic discount rate over the time interval $[3,7]$?

Question 2: What is the price of a pure discount bond with a face value of $1,000 purchased at the end of year 3 and matures at the end of year 7?

Bond Price Equation

In Part I we defined the variable r_u to be the random short rate of interest at time u. The equation for the expected short rate at time u from the perspective of time zero is [1]

$$\mathbb{E}\left[r_u\right] = r_\infty + \mathrm{Exp}\left\{-\lambda u\right\}(r_0 - r_\infty) \tag{3}$$

In Part II we defined the variable $R_{s,t}$ to be the stochastic discount rate over the time interval $[s,t]$. The equation for the stochastic discount rate is... [2]

$$R_{s,t} = \int_s^t r_u\,\delta u \tag{4}$$

In Part II we defined the variable $P(s,t)$ to be the price at time s of a pure discount bond that matures at time t and pays one dollar at maturity. The price of this pure discount bond may be written as the expectation of the path integral of the short rate over the time interval $[s,t]$. Using Equation (4) above the equation for bond price is... [2]

$$P(s,t) = \mathbb{E}\left[\mathrm{Exp}\left\{-\int_s^t r_u\,\delta u\right\}\,\bigg|\,I_s\right] = \mathbb{E}\left[\mathrm{Exp}\left\{-R_{s,t}\right\}\,\bigg|\,I_s\right] \tag{5}$$

We will define the variable Z to be a normally-distributed random variable with mean equal to Equation (1) above and variance equal to Equation (2) above. Given this definition of z we can rewrite bond price Equation (5) above as...

$$P(s,t) = \mathbb{E}\left[\mathrm{Exp}\left\{-z\right\}\right] \text{ ...where... } z \sim N\left[m,v\right] \tag{6}$$

Note that bond price in Equation (6) above is equal to the expected value of the exponential of negative z. We therefore need to calculate that expectation to get a bond price equation in closed-form.

The Expectation of a Normally-Distributed Random Variate

We are given the function $g(z)$ where the variable z is a normally-distributed random variate with mean m and variance v. Since z is normally-distributed the expectation of $g(z)$ is...

$$\mathbb{E}\left[g(z)\right] = \int_{-\infty}^{\infty} g(z)\,f(z)\,\delta z \tag{7}$$

In the function above $f(z)$ is the probability density function (PDF) of the normal distribution. The equation for the PDF is...

$$f(z) = \frac{1}{\sqrt{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}(z-m)^2\right\} \text{ ...where... m = mean of z ...and... v = variance of z} \tag{8}$$

We will now calculate the expectation of the normally-distributed random variate z in bond price Equation (6) above...

Calculating the Expectation

Because the random variable in Equation (6) is normally distributed we can rewrite Equation (6) as...

$$\begin{aligned}\mathbb{E}\left[\mathrm{Exp}\left\{-z\right\}\right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,V}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}(z-m)^2\right\}\mathrm{Exp}\left\{-z\right\}\delta z \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,v}}\,\mathrm{Exp}\left\{-\frac{1}{2\,v}(z^2-2\,m\,z+m^2+2\,v\,z)\right\}\delta z \end{aligned} \tag{9}$$

Noting that...

$$(z-m+v)^2 = z^2 - 2\,m\,z + 2\,v\,z + m^2 - 2\,m\,v + v^2 \tag{10}$$

We can rewrite Equation (9) above as...

$$\begin{aligned}
\mathbb{E}\Big[\mathrm{Exp}\Big\{-z\Big\}\Big] &= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}((z-m+v)^2+2\,m\,v-v^2)\Big\}\,\delta z \\
&= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}\,(z-m+v)^2\Big\}\mathrm{Exp}\Big\{-\frac{1}{2v}(2\,m\,v-v^2)\Big\}\,\delta z \\
&= \mathrm{Exp}\Big\{-\frac{1}{2v}\,(2\,m\,v-v^2)\Big\}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}\,(z-m+v)^2\Big\}\,\delta z \\
&= \mathrm{Exp}\Big\{-m+\frac{1}{2}\,v\Big\}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}\,(z-m+v)^2\Big\}\,\delta z \qquad (11)
\end{aligned}$$

We will define the constant α to be...

$$\alpha = m - v \qquad (12)$$

Using the definition in Equation (12) above we can rewrite Equation (11) as...

$$\mathbb{E}\Big[\mathrm{Exp}\Big\{-z\Big\}\Big] = \mathrm{Exp}\Big\{-m+\frac{1}{2}\,v\Big\}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}\,(z-\alpha)^2\Big\}\,\delta z \qquad (13)$$

Noting that...

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\,\mathrm{Exp}\Big\{-\frac{1}{2v}\,(z-\alpha)^2\Big\}\,\delta z = 1 \qquad (14)$$

We can rewrite Equation (13) as...

$$\mathbb{E}\Big[\mathrm{Exp}\Big\{-z\Big\}\Big] = \mathrm{Exp}\Big\{-m+\frac{1}{2}\,v\Big\} \qquad (15)$$

Using Equations (6) and (15) above the price at time zero of a pure discount bond that matures at time T and pays one dollar at maturity given the information set available at time zero (I_0) can be written as...

$$P(s,t) = \mathbb{E}\Big[\mathrm{Exp}\Big\{-z\Big\}\Big] = \mathrm{Exp}\Big\{-m+\frac{1}{2}\,v\Big\} \qquad (16)$$

Answers To Our Hypothetical Problem

Using expected short rate Equation (3) and the model parameters above the expected short rate at the beginning of the time interval $[3,7]$ is...

$$\mathbb{E}\Big[r_3\Big] = 0.09 + \mathrm{Exp}\Big\{-0.35\times 3\Big\}\times(0.04-0.09) = 0.0725 \qquad (17)$$

Question 1: What is the mean and variance of the stochastic discount rate over the time interval $[3,7]$?

Using Equations (1) and (17) above and the assumptions table above the stochastic discount rate mean over the time interval $[3,7]$ is...

$$\begin{aligned}
m &= 0.09\times(7-3) + (0.09-0.0725)\times\Big(\mathrm{Exp}\Big\{-0.35\times(7-3)\Big\}-1\Big)\times 0.35^{-1} \\
&= 0.32234 \qquad (18)
\end{aligned}$$

Using Equation (2) and the assumptions table above the variance of the stochastic discount rate variance over the time interval $[3,7]$ is...

$$
\begin{aligned}
v &= \frac{0.03^2}{2 \times 0.35^3} \times \left(2 \times 0.35 \times (7-3) - 3 + 4 \times \mathrm{Exp}\left\{ -0.35 \times (7-3) \right\} - \mathrm{Exp}\left\{ -2 \times 0.35 \times (7-3) \right\} \right) \\
&= 0.00762
\end{aligned}
\tag{19}
$$

Question 2: What is the price of a pure discount bond with a face value of \$1,000 purchased at the end of year 3 and matures at the end of year 7?

Using Equation (16) above bond price at time 3 is...

$$
P(3,7) = 1,000 \times \mathrm{Exp}\left\{ -0.32234 + \frac{1}{2} \times 0.00762 \right\} = 727.22 \tag{20}
$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Discount Rate,* February, 2013.